Quivers, varieties, and all that

Vertex functions of type D Nakajima quiver varieties

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Outline

What are we studying?

We study...

- certain natural generating functions ("vertex functions"),
- of algebro-geometric objects ("varieties"),
- arising from some combinatorial data ("quiver").

Specifically, we study the "type D" quivers, among type ADE, which arise from some natural classification.

Goal: find a nice product formula for these generating functions.

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Why do we care?

- Sits at the intersection of various areas of math:
  - Algebraic geometry
  - Representation theory
  - Mathematical physics
  - (Algebraic) combinatorics
- Helps us understand complex algebro-geometric structures with an analytic lens.

What is a quiver?

- A quiver is just a fancy name for a directed graph.
- It has vertices (dots) and arrows connecting the vertices.
- It's a skeleton for more complex algebraic and geometric structures.

Let's take a look at an example.

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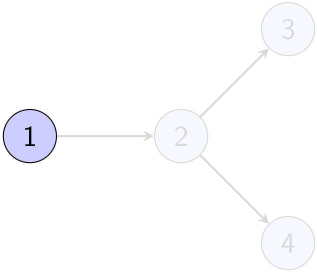
Example: D_4 Quiver



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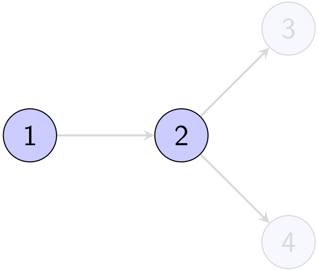

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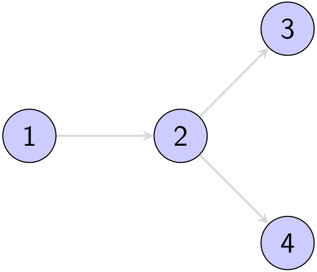

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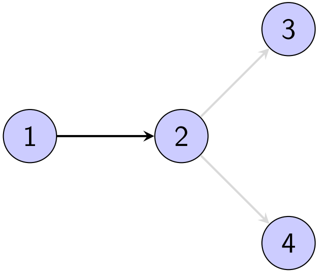

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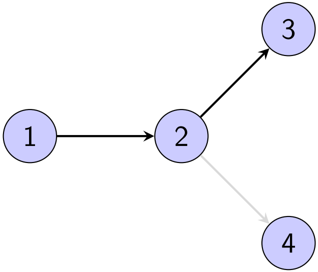

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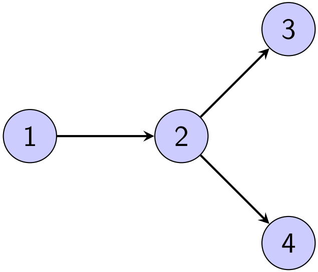

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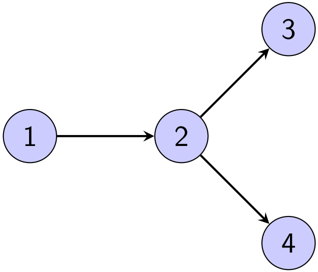

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D_n quivers

Definition

A D_n quiver is a quiver that has n vertices and $n-1$ edges, and has a Y-shaped branch at the end (its structure resembles a forked tree).

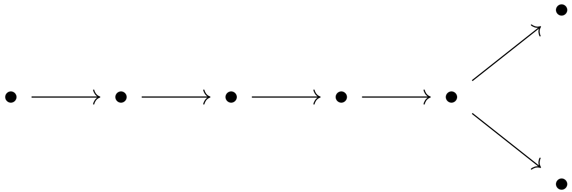

Figure: D_7 quiver

Quiver Representations

With this combinatorial object, we can build a representation out of it, called a *quiver representation*. A quiver representation assigns:

- A vector space (a space with “linear” structure) to each vertex,
- A linear map (a function that transforms one “linear“ structure to another) to each arrow.

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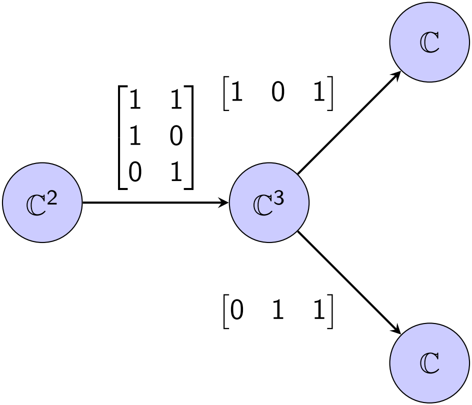
Example: D_4 Quiver Representation



What is a Nakajima Quiver Variety?

- Created by Hiraku Nakajima in 1994 [Nak94]
- A special space built from quiver representations
- Involves some fancy math like moment maps and Geometric Invariant Theory (GIT).

It's a way to study quiver representations geometrically.

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Hiraku Nakajima

- Japanese mathematician
- Born in 1962
- Introduced Nakajima quiver varieties in 1994 [Nak94]
- Awarded the Cole Prize in Algebra in 2003
- Current IMU President

Constructing a Nakajima Quiver Variety

Here's the procedure:

1. Start with a quiver
2. Frame it (add new vertices)
3. Double it (add reverse arrows)
4. Take representations of this new quiver
5. Apply some conditions (moment map, stability)
6. Take a quotient ("GIT quotient")

We get a geometric space with nice properties.

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Andrei Okounkov

- Russian-American mathematician
- Born in 1969
- Introduced vertex functions [Oko17]
- Fields Medal recipient in 2006

Example: D_1 Vertex Function

For the simplest quiver D_1 (single vertex, no arrows):

- The vertex function:

$$V(z) = \sum_{k \geq 0} \frac{(\hbar)_k}{(q)_k} z^k$$

- This is a well-known function: the q-binomial series.

Even in this simplest case, we see nontrivial structure.

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Type A Case

Theorem (Dinkins-Smirnov)

For type A Nakajima quiver varieties with certain restrictions, the vertex function $V(\boldsymbol{z})$ can be written as a product:

$$V(\boldsymbol{z}) = \prod F(\textit{some terms}),$$

where $F(\cdot)$ are the q-binomial series.

They used vertex operators Γ_- and Γ_+ to prove the theorem.

Main Theorem

Theorem (Dinkins-J.)

For type D Nakajima quiver varieties with certain restrictions, the vertex function $V(\boldsymbol{z})$ can be written as a product:

$$V(\boldsymbol{z}) = \prod_{\alpha \in \boldsymbol{\Phi}^-} F(\text{some terms}),$$

where $F(\cdot)$ are the q-binomial series.

The proof uses Macdonald polynomial theory.

What Does This Mean?

- Simple product formula for a complicated object
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Concrete Computations

For D_4 quiver with $\mathrm{v} = (1, 2, 1, 1)$ and $\mathrm{w} = (0, 0, 0, 1)$:

$$V(z_1, z_2, z_3, z_4) = F((q/\hbar)z_2) \cdot F(z_2 z_3) \cdot F((q/\hbar)^2 z_1 z_2) \\ \cdot F((q/\hbar)z_1 z_2 z_3) \cdot F(z_1 z_2^2 z_3 z_4)$$

Each factor corresponds to a specific root in the D_4 root system.

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Acknowledgments

The author owes his deepest gratitude to:

- Dr. Hunter Dinkins (mentor)
- Prof. Alexei Borodin
- MIT PRIMES-USA program

Any questions?

Quiver Representation

A **representation** of a quiver $Q = (I, E)$ is a tuple of data

$$((V_i)_{i \in I}, (X_e)_{e \in E})$$

where:

- V_i are complex vector spaces
- $X_e \in \mathrm{Hom}(V_{o(e)}, V_{i(e)})$ are linear maps
- $o(e)$ and $i(e)$ denote the outgoing and incoming vertices of edge e, respectively

Nakajima Quiver Variety

A **Nakajima quiver variety** is defined as:

$$\mathcal{M}_Q(\mathrm{v}, \mathrm{w}) := \mu^{-1}(0)^s / G_\mathrm{v}$$

where:

- μ is the moment map
- $\mu^{-1}(0)^s$ is the set of stable points in $\mu^{-1}(0)$
- $G_\mathrm{v} := \prod_{i \in I} \mathrm{GL}(\mathrm{v}_i, \mathbb{C})$
- v and w are dimension vectors

Vertex Function

The **vertex function** $V(\boldsymbol{z})$ of a Nakajima quiver variety $\mathcal{M}$ is defined as:

$$V(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{\infty,*}\left(\hat{\mathcal{O}}_{\mathrm{vir}}^{\boldsymbol{d}}\right)\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(\mathcal{M})_{loc}[[z]]$$

where:

- $\mathrm{ev}_{\infty,*}$ is an evaluation map
- $\hat{\mathcal{O}}_{\mathrm{vir}}^{\boldsymbol{d}}$ is the symmetrized virtual structure sheaf
- $\boldsymbol{z}^{\boldsymbol{d}}$ are formal variables
- $K_{\mathsf{T}\times\mathbb{C}_q^\times}(\mathcal{M})_{loc}$ is the localized equivariant K-theory

Root System

A **root system** Φ in a Euclidean space E is a finite set of non-zero vectors (called roots) that satisfy:

- The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α and $-\alpha$
- For every root $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant

where σ_α is defined by:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

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